Performance Modelling and Evaluation of a Wireless Cellular Network of Voice Calls

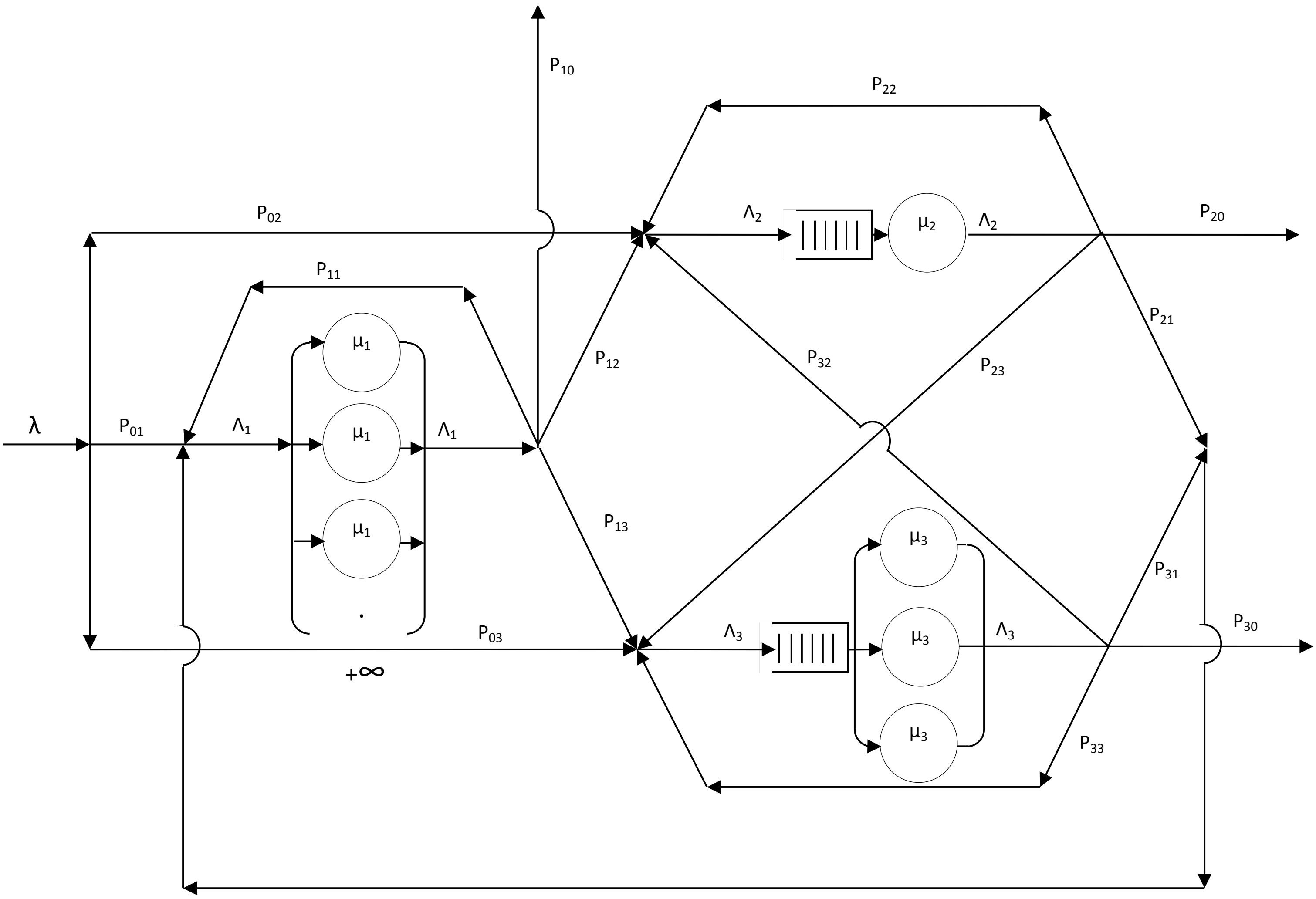
1.0 INTRODUCTION

This coursework analysed wireless cellular network calls. This consisted of three wireless cells: a delay station, one queuing station with one server and another queuing station with three servers. The performance parameters for the exponential open queuing networks and open GE-type network were obtained and the impact of the square coefficient of variation was observed.

1.1 Analysis of an open queuing network

The figure below shows the open queuing network with P_{ij}, ij ϵ (0,1,2,3) as the transition probabilities in the network. Λ_1, Λ_2 and Λ_3 are the mean flow rates while μ_1,μ_2 and μ_3 are the mean service rates at stations 1, 2 and 3 respectively. The arrival and service rate follows the Poisson process and hence Jackson's theorem was applied to the network.

Figure 1: Wireless Cellular Open Exponential Queuing Network



From the above transition probabilities, the new calls are P_{01}, P_{02}, and P_{03}. The handoff calls are P_{21}, P_{31}, P_{23}, and P_{32}. The continuing calls are P_{22}, P_{33}, and P_{11}. The terminating calls are P_{10}, P_{20}, and P_{30}.

At a steady state, flow rate in = flow rate out.

The flow balance equations for the overall mean arrival rates are as shown below:

$$\Lambda_1 = P_1 \lambda + P_{11} \Lambda_1 + P_{21}\Lambda_2 + P_{31}\Lambda_3 \quad (1)$$

$$\Lambda_2 = P_2 \lambda + P_{12} \Lambda_1 + P_{22}\Lambda_2 + P_{32}\Lambda_3 \quad (2)$$

$$\Lambda_3 = P_3 \lambda + P_{13} \Lambda_1 + P_{23}\Lambda_2 + P_{33}\Lambda_3 \quad (3)$$

This implies

$$(1 - P_{11}) \Lambda_1 - P_{21}\Lambda_2 - P_{31}\Lambda_3 = P_1 \lambda$$

$$- P_{12} \Lambda_1 + (1 - P_{22})\Lambda_2 - P_{32}\Lambda_3 = P_2 \lambda$$

$$- P_{13} \Lambda_1 - P_{23}\Lambda_2 + (1 - P_{33})\Lambda_3 = P_3 \lambda$$

Using Cramer's rule,

$$\Lambda_1 = \frac{\begin{vmatrix} \lambda P_1 & -P_{21} & -P_{31} \\ \lambda P_2 & 1-P_{22} & -P_{32} \\ \lambda P_3 & -P_{23} & 1-P_{33} \end{vmatrix}}{\begin{vmatrix} 1-P_{11} & -P_{21} & -P_{31} \\ -P_{12} & 1-P_{22} & -P_{32} \\ -P_{13} & -P_{23} & 1-P_{33} \end{vmatrix}}, \quad \Lambda_2 = \frac{\begin{vmatrix} 1-P_{11} & \lambda P_1 & -P_{31} \\ -P_{12} & \lambda P_2 & -P_{32} \\ -P_{13} & \lambda P_3 & 1-P_{33} \end{vmatrix}}{\begin{vmatrix} 1-P_{11} & -P_{21} & -P_{31} \\ -P_{12} & 1-P_{22} & -P_{32} \\ -P_{13} & -P_{23} & 1-P_{33} \end{vmatrix}}$$

$$\Lambda_3 = \frac{\begin{vmatrix} 1-P_{11} & -P_{21} & \lambda P_1 \\ -P_{12} & 1-P_{22} & \lambda P_2 \\ -P_{13} & -P_{23} & \lambda P_3 \end{vmatrix}}{\begin{vmatrix} 1-P_{11} & -P_{21} & -P_{31} \\ -P_{12} & 1-P_{22} & -P_{32} \\ -P_{13} & -P_{23} & 1-P_{33} \end{vmatrix}}$$

Solving the determinants gives:

$$\Lambda_1 = \frac{P_{01}\lambda\begin{vmatrix} 1-P_{22} & -P_{32} \\ -P_{23} & 1-P_{33} \end{vmatrix} - P_{02}\lambda\begin{vmatrix} -P_{21} & -P_{31} \\ -P_{23} & 1-P_{33} \end{vmatrix} + P_{03}\lambda\begin{vmatrix} -P_{21} & -P_{31} \\ 1-P_{22} & -P_{32} \end{vmatrix}}{1-P_{11}\begin{vmatrix} 1-P_{22} & -P_{32} \\ -P_{23} & 1-P_{33} \end{vmatrix} - (-P_{12})\begin{vmatrix} -P_{21} & -P31 \\ -P_{23} & 1-P_{33} \end{vmatrix} + (-P_{13})\begin{vmatrix} -P_{21} & -P_{31} \\ 1-P_{22} & -P_{32} \end{vmatrix}}$$

$$\Longrightarrow \quad \frac{\lambda[P_{01}(1-P_{22})(1-P_{22})-P_{01}P_{23}P_{32}+P_{02}P_{21}(1-P_{33})+P_{02}P_{23}P_{31}+P_{03}P_{21}P_{31}(1-P_{22})]}{(1-P_{11})(1-P_{22})(1-P_{33})-P_{23}P_{32}(1-P_{11})-P_{12}P_{21}(1-P_{33})-P_{12}P_{32}P_{13}+P_{31}P_{12}P_{23}+P_{31}P_{13}(1-P_{22})}$$

$$\Lambda_2 = \frac{1-P_{11}\begin{vmatrix} P_{02}\lambda & -P_{32} \\ P_{03}\lambda & 1-P_{33} \end{vmatrix} - (-P_{12})\begin{vmatrix} P_{01}\lambda & -P_{31} \\ P_{03}\lambda & 1-P_{33} \end{vmatrix} + (-P_{13})\begin{vmatrix} P_{01}\lambda & -P_{31} \\ P_{02}\lambda & -P_{32} \end{vmatrix}}{1-P_{11}\begin{vmatrix} 1-P_{22} & -P_{32} \\ -P_{23} & 1-P_{33} \end{vmatrix} - (-P_{12})\begin{vmatrix} -P_{21} & -P_{31} \\ -P_{23} & 1-P_{33} \end{vmatrix} + (-P_{13})\begin{vmatrix} -P_{21} & -P_{31} \\ 1-P_{22} & -P_{32} \end{vmatrix}}$$

$$= \frac{\lambda[P_{02}(1-P_{33})(1-P_{11})+P_{32}P_{02}(1-P_{11})+P_{12}P_{01}(1-P_{33})+P_{03}P_{31}P_{12}+P_{13}P_{32}P_{01}-P_{13}P_{02}P_{13}]}{(1-P_{11})(1-P_{22})-P_{23}P_{32}(1-P_{11})-P_{12}P_{21}(1-P_{33})-P_{12}P_{21}(1-P_{33})-P_{12}P_{23}P_{31}-P_{13}P_{21}P_{32}-P_{13}P_{31}(1-P_{22})}$$

$$\Lambda_3 = \frac{1-P_{11}\begin{vmatrix} 1-P_{22} & P_{02}\lambda \\ -P_{23} & P_{03}\lambda \end{vmatrix} - (-P_{12})\begin{vmatrix} -P_{21} & P_{01}\lambda \\ -P_{23} & P_{03}\lambda \end{vmatrix} + (-P_{13})\begin{vmatrix} -P_{21} & P_{01}\lambda \\ 1-P_{22} & P_{02}\lambda \end{vmatrix}}{1-P_{11}\begin{vmatrix} 1-P_{22} & -P_{32} \\ -P_{23} & 1-P_{33} \end{vmatrix} - (-P_{12})\begin{vmatrix} -P_{21} & -P_{31} \\ -P_{23} & 1-P_{33} \end{vmatrix} + (-P_{13})\begin{vmatrix} -P_{21} & -P_{31} \\ 1-P_{22} & -P_{32} \end{vmatrix}}$$

$$\frac{[P_{03}(1-P_{22})(1-P_{11})+P_{23}P_{02}(1-P_{11})-P_{21}P_{12}P_{03}+P_{12}P_{23}P_{01}+P_{21}P_{13}P_{02}+P_{13}P_{01}(1-P_{22})}{(1-P_{11})(1-P_{22})(1-P_{33})-P_{32}P_{23}(1-P_{11})-P_{12}P_{21}(1-P_{33})-P_{12}P_{23}P_{31}-P_{13}P_{21}P_{32}-P_{13}P_{21}P_{32}-P_{31}P_{31}(1-P_{22})}$$

Calculating the channel utilisations at stations 1, 2 and 3:

$\rho = \lambda/c\mu$ was used where

ρ = utilisation

λ = mean arrival rate

μ = mean service rate

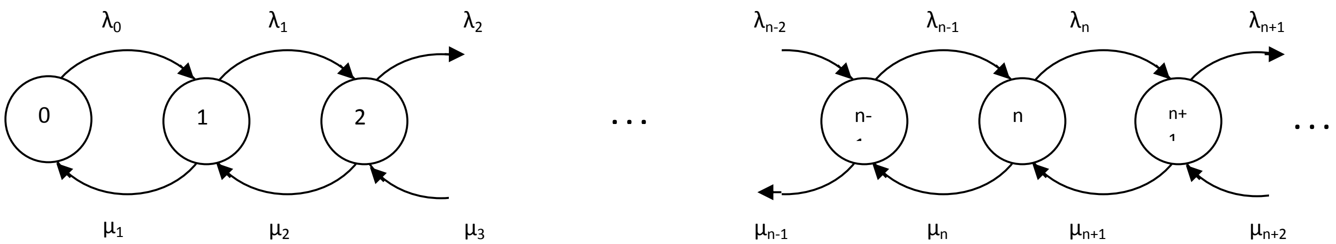
c = number of servers

$\boldsymbol{\rho_1} = \frac{\boldsymbol{\Lambda_1}}{\infty\boldsymbol{\mu_1}} = \mathbf{0}$, $\boldsymbol{\rho_2} = \frac{\boldsymbol{\Lambda_2}}{\boldsymbol{\mu_2}}$, and $\boldsymbol{\rho_3} = \frac{\boldsymbol{\Lambda_3}}{3\boldsymbol{\mu_3}}$

The mean network throughput $\boldsymbol{\lambda_T}$ is

$$\boldsymbol{\lambda_T = P_{10}\Lambda_1 + P_{20}\Lambda_1 + P_{30}\Lambda_3}$$

Figure 2: The general birth and death model.



At a steady state, flow rate in = flow rate out.

For n = 0, $\lambda_0 P_0 = \mu_1 P_1$ ==> $P_1 = \frac{\lambda_0}{\mu_1} P_0$

For n = 1, $P_1\lambda_1 = P_2\mu_2$ ==> $P_2 = \frac{\lambda_1}{\mu_2} P_1$ ==> $P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2} P_0$

For n = 3, $P_2\, \lambda_2 = P_3\mu_3$ ==> $P_3 = \frac{\lambda_2}{\mu_3} P_2$ ==> $P_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} P_0$

For any value of n, $P_n = \frac{\lambda_0\lambda_1 \ldots \lambda_{n-1}}{\mu_1\mu_2 \ldots \mu_n} P_0$

Let $C_n = \frac{\lambda_0\lambda_1 \ldots \lambda_{n-1}}{\mu_1\mu_2 \ldots \mu_n}$

then $P_n = C_n P_0$

But

$\sum_{i=0}^{n} P_i = 1$ or $P_0 + P_1 + P_2 + \ldots\ldots = 1$

=> $P_0 + C_1P_0 + C_2P_2 + \ldots\ldots = 1$ ==> $P_0 = [\,1 + C_1 + C_2 + C_3 + \ldots\,]^{-1}$

Therefore, $P_0 = [1 + \sum_{n=1}^{\infty} C_n\ ]^{-1}$

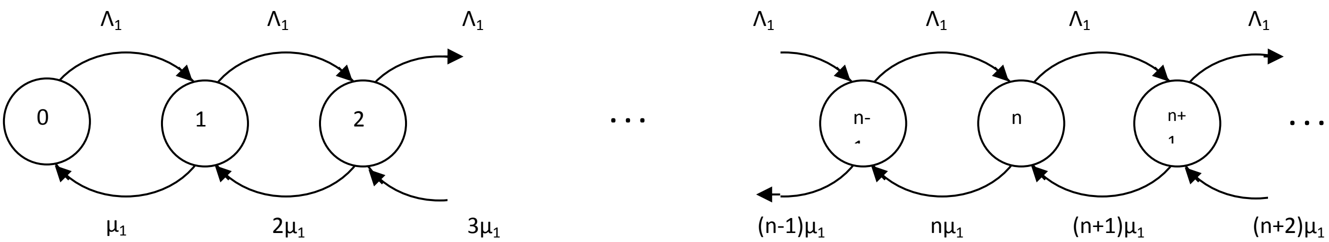
According to Jackson, if we let $P(n_1, n_2, n_3, \ldots\ldots)$ denote the steady state probability that they are n_i customers in ith node(i= 1,2,3) and if $\Lambda_i < c_i\mu_i$

then,

$P(n_1, n_2, n_3) = P_1(n_1)\, P_2(n_2)\, P_3(n_3)$. This means that the systems can be decomposed into individual queuing systems and analysed independently. The analysis of the individual queuing system is shown below.

1.2 Delay Station I

Figure 3: The transition diagram for M/M/∞ at delay station 1



The delay station 1 still behaves as a single server since distribution of the minimum mean remaining service time in the servers is exponential.

Hence, $\lambda_n = \Lambda_1$, and $\mu_n = n\mu_1$ ==> $C_n = \frac{\Lambda_1^n}{n!\mu_1^n} = \frac{u_1^n}{n!}$ where $u_1 = \frac{\Lambda_1}{\mu_1}$ for n = 1, 2, 3, . . .

$P_0 = [1 + \sum_{n=1}^{\infty} \frac{u_1^n}{n!}]^{-1}$ from 2.9. But, $1 + \sum_{n=1}^{\infty} \frac{u_1^n}{n!} = 1 + u_1 + \frac{u_1^2}{2!} + \frac{u_1^3}{3!} + \frac{u_1^4}{4!} + \ldots$ $= e^{u_1}$.

$P_0 = e^{-u_1}$ ==> $\boldsymbol{P_1(n_1) = e^{-u_1} (\frac{u_1^n}{n!})}$

The mean residence time in the station is given by:

$W_1 = W_q + W_s = W_s$ $W_q = 0$ since there is no queue.

But, $W_s = 1/\mu_1 = E(s_1)$; that's $W_s = \frac{1}{\mu_1(1-\rho)}$ since $\rho = 0$, then $W_s = 1/\mu_1$.

Similarly, $L_1 = L_q + L_s = L_s$, since $L_q = 0$.

From Little's formula, $L_s = \Lambda_1 W_s = \Lambda_1/\mu_1 = u_1$

Therefore $L_1 = u_1$

$W_1^* = W_1 V_1$ where V_1 = visiting ratio at Station I.

From the forced flow law, where X_i is the throughput of the device i, this can be represented by Λ_i. V_i is the mean number of visits (requests) at the device per job and X_o is the mean throughput of the network.

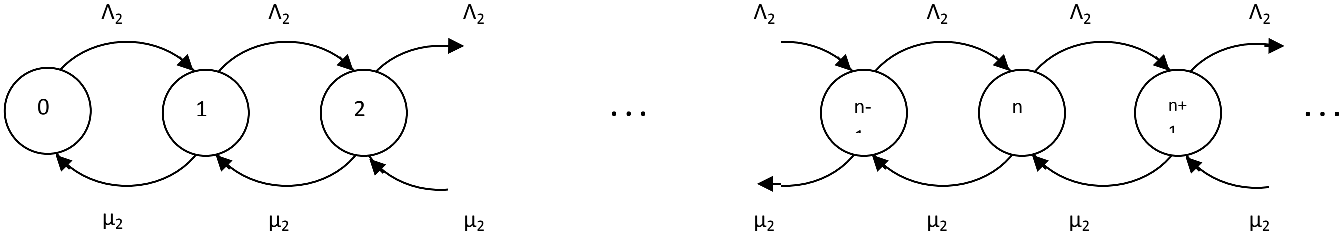
$Xi = V_i Xo$ ==> $V_1 = \frac{\Lambda_1}{\lambda_T}$, where $X_1 =$ V_1 , $X_o = \Lambda_1$

Therefore, $W_1^* = W_1 \frac{\Lambda_1}{\lambda_T} = \frac{L_1}{\lambda_T}$, since $W_1 \Lambda_1 = L_1$

$$\boldsymbol{W_1^* = \frac{u_1}{\lambda_T}}$$

1.3 Queuing Station II (M/M/1)

Figure 4. The transition diagram of station II, which is an example of an M/M/1 queuing model.



Here the $\lambda_{in} = \Lambda_2$ and $\mu_n = \mu_2$

From the birth and death model, $C_n = \frac{\Lambda_2^n}{\mu_2^n} = \rho_2^n$.

$P_0 = [1 + \sum_{n=1}^{\infty} \rho_2^n]^{-1}$ at steady state $\rho_2 < 1$.

From the geometric series, $[1 + \sum_{n=1}^{\infty} \rho_2^n] = 1 + \rho_2 + \rho_2^2 + \rho_2^3 + \rho_2^4 + \ldots$ $= \frac{1}{1-\rho_2}$

$\Longrightarrow P_0 = 1 - \rho_2$

Therefore: $\boldsymbol{P_2(n_2) = (1 - \rho_2)\,\rho_2^n}$

The mean number of calls in the station is given by:

$L_2 = E[N_2] = \sum_{n=0}^{\infty} n\rho_2^n P_0 = P_0\rho_2 \sum_{n=0}^{\infty} n\rho_2^{n-1} = P_0\rho_2 \sum_{n=0}^{\infty} \frac{d}{d\rho_2}\rho_2^n$

$P_0\rho_2 \frac{d}{d\rho_2}\sum_{n=0}^{\infty} \rho_2^n = P_0\rho_2 \frac{d}{d\rho_2}(1 + \rho_2 + \rho_2^2 + \rho_2^3 + \rho_2^4 + \ldots)$

$= P_0\rho_2 \frac{d}{d\rho_2}\left(\frac{1}{1-\rho_2}\right) = \frac{P0\rho_2}{(1-\rho_2)^2} = \frac{(1-\rho_2)\rho_2}{(1-\rho_2)^2}$

Therefore: $\mathbf{L_2} = \frac{\rho_2}{(1-\rho_2)}$

From Little's Law,

$L_2 = \Lambda_2 W_2 \Longrightarrow W_2 = \frac{L_2}{\Lambda_2} = \frac{\rho_2}{\Lambda_2(1-\rho_2)} = \frac{1}{\mu_2(1-\rho_2)}$

Therefore: $\mathbf{W_2} = \frac{E(s_2)}{(1-\rho_2)}$

$W_2^* = W_2 V_2$ where V_2 = visiting ratio at Station II.

Applying the forced flow law:

$\Lambda_2 = V_2\,\lambda_T \Longrightarrow V_2 = \frac{\Lambda_2}{\lambda_T}$, therefore $W_2^* = W_2 \frac{\Lambda_2}{\lambda_T} = \frac{L_2}{\lambda_T}$

$\mathbf{W_2^*} = \frac{\rho_2}{\lambda_T(1-\rho_2)}$

1.4 Queuing Station III

This is an M/M/C queuing system with three servers. The general death and birth process can also be used to analyse the system.

Here:

$\lambda_n = \Lambda_3$

and $\mu_n = \begin{cases} n\mu_3 & \text{for } n = 0, 1, 2, \ldots, c-1 \\ c\mu_3 & \text{for } n = c, c+1, c+2, \ldots \end{cases}$

From the general birth and death model, $C_n =$

$$\begin{cases} \frac{\Lambda_3^n}{n!\,\mu_3^n} & \text{for } n = 1, 2, 3, \ldots, c-1 \\ c!\,c^{n-c}\frac{\Lambda_3^n}{\mu_3^n} & \text{for } n = c, c+1, c+2, \ldots \end{cases}$$

==> $P_n =$

$$\begin{cases} \frac{\Lambda_3^n}{n!\,\mu_3^n} P_0 & \text{for } n = 1, 2, 3, \ldots, c-1 \\ \frac{\Lambda_3^n}{c!c^{n-c}\mu_3^n} P_0 & \text{for } n = c, c+1, c+2, \ldots \end{cases}$$

Let $\frac{\Lambda_3^n}{\mu_3^n}$ = u, then

$$P_n = \begin{cases} \frac{u_3^n}{n!} P_0 & \text{for } n = 1, 2, 3, \ldots, c-1 \\ \frac{\rho_3^n}{c!c^{-c}} P_0 & \text{for } n = c, c+1, c+2, \ldots \end{cases}$$

By summation,

$P_0 + P_1 + P_2 + P_3 + \ldots = 1$ ==> $\sum_{n=0}^{c-1} \frac{u_3^n}{n!} P_0 + \sum_{n=c}^{\infty} \frac{\rho_3^n}{c!c^{-c}} P_0 = 1$

therefore, $P_0 = \left[\sum_{n=0}^{c-1} \frac{u_3^n}{n!} + \sum_{n=c}^{\infty} \frac{\rho_3^n}{c!c^{-c}}\right]^{-1}$, but $\sum_{n=c}^{\infty} \frac{\rho_3^n}{c!c^{-c}} =$

$$\sum_{n=c}^{\infty} \frac{u_3^c \rho_3^{n-c}}{c!} = \frac{u_3^c}{c!}\sum_{n=c}^{\infty} \frac{\rho_3^{n-c}}{c!} = \frac{u_3^c}{c!}\sum_{k=0}^{\infty} \rho_3^k = \frac{u_3^c}{c!}\left(\frac{1}{1-\rho_3}\right)$$

For this process, c = 3.

===> $\sum_{n=0}^{c-1} \frac{u_3^n}{n!} = \sum_{n=0}^{2} \frac{u_3^n}{n!} = 1 + u_3 + \frac{u_3^2}{2!}$

Therefore: $P_0 = \left[1 + u_3 + \frac{u_3^2}{2} + \frac{u_3^3}{6}\left(\frac{1}{1-\rho_3}\right)\right]^{-1}$

then:

$$\boldsymbol{P_n} = \begin{cases} \frac{u_3^n}{n!}\left[1 + u_3 + \frac{u_3^2}{2} + \frac{u_3^3}{6}\left(\frac{1}{1-\rho_3}\right)\right]^{-1} & \textbf{for } n = 0,\ 1,\ 2,\ 3 \\ \frac{9\rho_3^n}{2}\left[1 + u_3 + \frac{u_3^2}{2} + \frac{u_3^3}{6}\left(\frac{1}{1-\rho_3}\right)\right]^{-1} & \textbf{for } n = 4,\ 5,\ 6\ \dots \end{cases}$$

$L_q = E[N_q] = \sum_{n=3}^{\infty}(n-3)P_n = \sum_{k=0}^{\infty} kP_{3+k}$, but $P_{k+3} = \frac{\rho_3^{3+k}}{3!3^{3-3}} P_0 = \frac{\rho_3^3\rho_3^k}{3!3^{3-3}} P_0 = \frac{u_3^3\rho_3^k}{3!}$ P_0

$L_q = \sum_{k=0}^{\infty} k \frac{u_3^3\rho_3^k}{3!} P_0 = P_0\frac{u_3^3}{3!}\sum_{k=0}^{\infty} k\rho_3^k = P_0\frac{u_3^3\rho_3}{3!}\sum_{k=0}^{\infty} k\rho_3^{k-1} =$ $P_0\frac{u_3^3\rho_3}{3!}\sum_{k=0}^{\infty}\frac{d}{d\rho}\rho_3^k$

Therefore: $L_q = P_0\frac{u_3^3\rho_3}{3!}\frac{d}{d\rho}\sum_{k=0}^{\infty}\rho_3^k = P_0\frac{u_3^3\rho_3}{3!}\frac{d}{d\rho}\left(\frac{1}{1-\rho_3}\right) = \frac{P_0u_3^3\rho_3}{6(1-\rho_3)^2}$

By Little's formula, $L_s = \Lambda_3 W_s$ ==> $L_s = \Lambda_3 / \mu_s = u_3$, but $L_3 = L_q + L_s$

$$\boldsymbol{L_3 = \frac{P_0u_3^3\rho_3}{6(1-\rho_3)^2} + u_3}$$

Applying Little's law to the station:

$L_3 = \Lambda_3 W_3$

==> $\boldsymbol{W_3 = \left(\frac{P_0u_3^3\rho_3}{6(1-\rho_3)^2} + u3\right) / \Lambda_3}$

$W_3^* = W_3 V_3 = W_3 \Lambda_3 / \lambda_T$, by forced flow law, where $\Lambda_3 = V_3 \lambda_T$.

Therefore: $\boldsymbol{W_3^* = L_3 / \lambda_T}$

The joint steady state probability is:

$P(n_1, n_2, n_3) = P1(n_1)P_2(n_2)P_3(n_3)$

==> $P(n_1, n_2, n_3)$ =

$$\begin{cases} e^{-u_1}\left(\frac{u_1^n}{n!}\right)(1-\rho_2)\rho_2^n\left[1+\ u_3\ +\ \frac{u_3^2}{2}\ +\ \frac{u_3^3}{6}\left(\frac{1}{1-\rho_3}\right)\right]^{-1}\frac{u_3^n}{n!} & for\ \ n=0,1,2 \\ e^{-u_1}\left(\frac{u_1^n}{n!}\right)(1-\rho_2)\rho_2^n\left[1+\ u_3\ +\ \frac{u_3^2}{2}\ +\ \frac{u_3^3}{6}\left(\frac{1}{1-\rho_3}\right)\right]^{-1}\frac{9\rho_3^n}{2} & for\ \ n=3,4,5,\ \ldots \end{cases}$$

and the residence time of the call is:

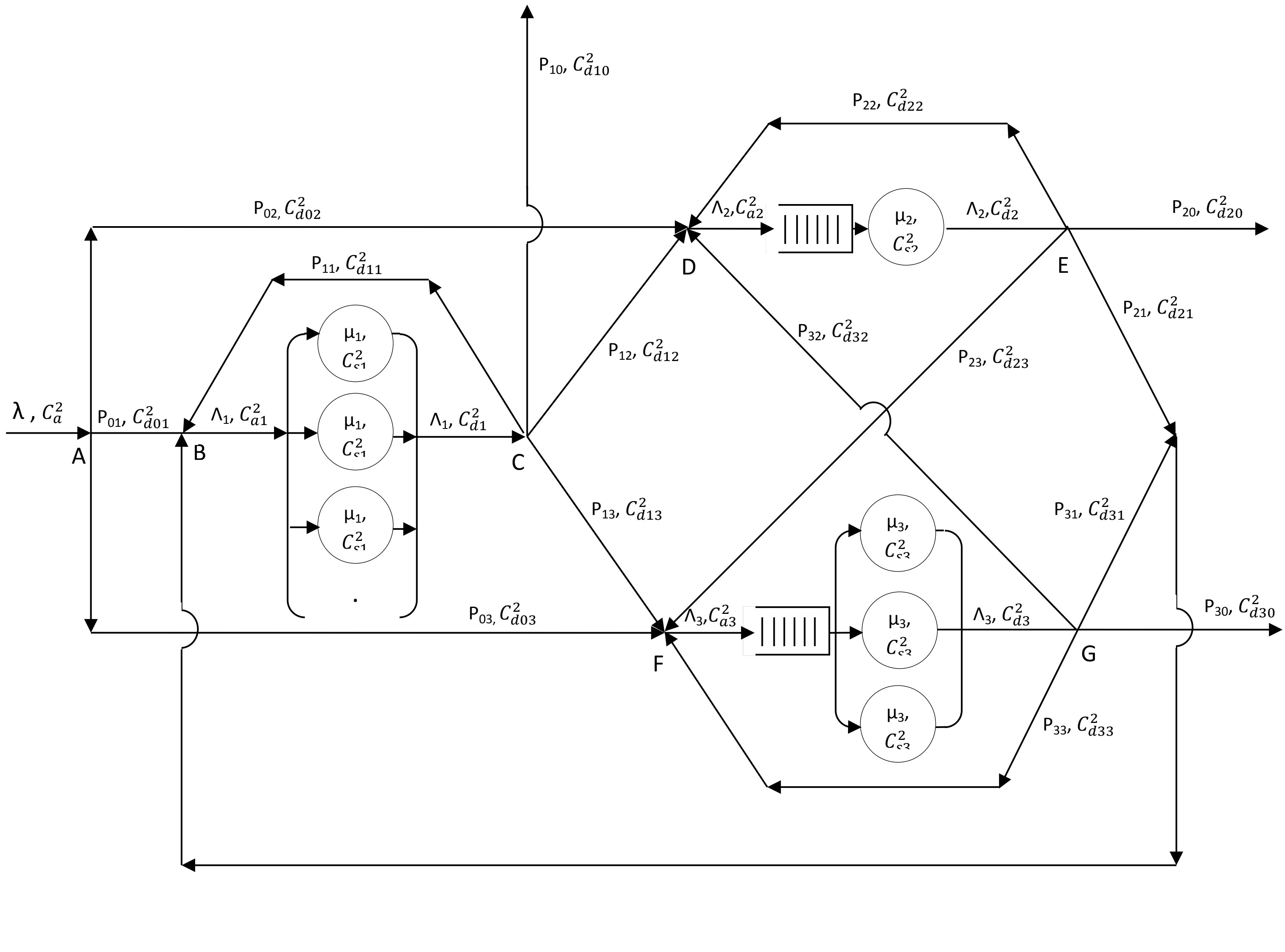
$W = W_1 + W_2 + W_3$ by substitution gives

$$W = E(s_1) + \frac{E(s_2)}{(1-\rho_2)} + \frac{\left(\frac{P_0 u_3^3 \rho_3}{6(1-\rho_3)^2} + u3\right)}{\Lambda 3}$$

2.0 PART II: ANALYSIS OF AN OPEN GE-TYPE QUEUING NETWORK

Here, the throughput of the system is the same as that of Part I because the arrival and service rates are independent of variance.

Figure 5: A Wireless Cellular Open GE-Type Queuing Network with labelled squared coefficients of variance



To analyse the GE/GE/∞ we consider the equation for the departure from a station to be given by:

$C_d^2 = (1 - \rho)\rho + (1 - \rho)\, C_a^2 + \rho^2\, C_s^2$

Therefore the departure delay at Queuing Station 1 is:

$C_{d1}^2 = (1 - \rho_1)\rho_1 + (1 - \rho_1)\, C_{a1}^2 + \rho_1^2\, C_{s1}^2$ but $\rho_1 = \frac{\Lambda_1}{\infty \mu_1} = 0$

==> $C_{d1}^2 = C_{a1}^2$

The departure at Queuing Station II is given by:

$C_{d2}^2 = (1 - \rho_2)\rho_2 + (1 - \rho_2)\, C_{a2}^2 + \rho_2^2\, C_{s2}^2$ where, $\rho_2 = \frac{\Lambda_2}{\mu_2}$

Similarly, the departure at Queuing Station III is given by:

$C_{d3}^2 = (1 - \rho_3)\rho_3 + (1 - \rho_3)\, C_{a3}^2 + \rho_3^2\, C_{s3}^2$ where $\rho_3 = \frac{\Lambda_3}{3\mu_3}$

In this network, there is splitting at point A, C, E and G and mergers at points B, D and F.

Splitting at A:

Splitting equation for a GE-Type process is:

$\lambda_i = P_i\, \lambda$, $C_{di}^2 = (1 - P_i) + P_i\, C_d^2$

then:

$\lambda_{01} = P_{01}\, \lambda$, $\lambda_{02} = P_{02}\, \lambda$, $\lambda_{03} = P_{03}\, \lambda$

and $C_{d01}^2 = (1 - P_{01}) + P_{01} C_a^2$, $C_{d02}^2 = (1 - P_{02}) + P_{02} C_a^2$, $C_{d03}^2 = (1 - P_{03}) + P_{03} C_a^2$

Splitting at C:

$\Lambda_{10} = P_{10}\, \Lambda_1$, $\Lambda_{11} = P_{11}\, \Lambda_1$, $\Lambda_{12} = P_{12}\, \Lambda_1$, $\Lambda_{13} = P_{13}\, \Lambda_1$

and $C_{d10}^2 = (1 - P_{10}) + P_{10} C_{d1}^2$

$C_{d11}^2 = (1 - P_{11}) + P_{11} C_{d1}^2$

$C_{d12}^2 = (1 - P_{12}) + P_{12} C_{d1}^2$

$$C_{d13}^2 = (1 - P_{13}) + P_{13}C_{d1}^2$$

Similarly, splitting at E:

$$\Lambda_{20} = P_{20}\ \Lambda_2,\ \ \Lambda_{21} = P_{21}\ \Lambda_2,\ \ \Lambda_{22} = P_{22}\ \Lambda_2,\ \ \Lambda_{23} = P_{23}\ \Lambda_2$$

and $C_{d20}^2 = (1 - P_{20}) + P_{20}C_{d2}^2$

$$C_{d21}^2 = (1 - P_{21}) + P_{21}C_{d2}^2$$

$$C_{d22}^2 = (1 - P_{22}) + P_{22}C_{d2}^2$$

$$C_{d23}^2 = (1 - P_{23}) + P_{23}C_{d2}^2$$

Splitting at G:

$$\Lambda_{30} = P_{30}\ \Lambda_3,\ \ \Lambda_{31} = P_{31}\ \Lambda_3,\ \ \Lambda_{32} = P_{32}\ \Lambda_3,\ \ \Lambda_{33} = P_{33}\ \Lambda_3$$

and $C_{d30}^2 = (1 - P_{30}) + P_{30}C_{d3}^2$

$$C_{d31}^2 = (1 - P_{31}) + P_{31}C_{d3}^2$$

$$C_{d32}^2 = (1 - P_{32}) + P_{32}C_{d3}^2$$

$$C_{d33}^2 = (1 - P_{33}) + P_{33}C_{d3}^2$$

Considering merging at B, D and F.

Given $\lambda = \sum_{i=1}^{n} \lambda_i$ and $C_d^2 = \left\{\frac{\lambda}{\sum_{i=1}^{n}\left(\frac{\lambda_i P_i}{C_{di}^2 + 1}\right)}\right\} - 1$

then, merging at B

$$\Lambda_1 = P_{21}\Lambda_2 + P_{31}\Lambda_3 + P_{11}\ \Lambda_1 + P_1\ \lambda$$

$$C_{a1}^2 = \left\{\frac{\Lambda_1}{\left(\frac{\lambda P_{01}}{C_{d01}^2 + 1}\right) + \left(\frac{\Lambda_1 P_{11}}{C_{d11}^2 + 1}\right) + \left(\frac{\Lambda_2 P_{21}}{C_{d21}^2 + 1}\right) + \left(\frac{\Lambda_3 P_{31}}{C_{d31}^2 + 1}\right)}\right\} - 1$$

and merging at D

$$\Lambda_2 = P_{22}\Lambda_2 + P_{32}\Lambda_3 + P_{12}\ \Lambda_1 + P_2\ \lambda$$

$$C_{a2}^2 = \left\{ \frac{\Lambda_2}{\left(\frac{\lambda P_{02}}{C_{d02}^2+1}\right) + \left(\frac{\Lambda_1 P_{12}}{C_{d12}^2+1}\right) + \left(\frac{\Lambda_2 P_{22}}{C_{d22}^2+1}\right) + \left(\frac{\Lambda_3 P_{32}}{C_{d32}^2+1}\right)} \right\} - 1$$

Merging at F:

$$\Lambda_3 = P_{13}\Lambda_1 + P_{23}\Lambda_2 + P_{33}\Lambda_3 + P_3 \lambda$$

$$C_{a3}^2 = \left\{ \frac{\Lambda_3}{\left(\frac{\lambda P_{03}}{C_{d03}^2+1}\right) + \left(\frac{\Lambda_1 P_{13}}{C_{d13}^2+1}\right) + \left(\frac{\Lambda_2 P_{23}}{C_{d23}^2+1}\right) + \left(\frac{\Lambda_3 P_{33}}{C_{d33}^2+1}\right)} \right\} - 1$$

The performance parameters at different stations are calculated in this way.

2.1 Delay Station I (GE/GE/∞)

It was observed that the mean waiting time at stations depended on the mean service time; the server is infinite and any customer would immediately get the requested service.

Therefore:

$$\boldsymbol{W_1 = E(s_1)}$$

By Little's formula, ==> $L_1 = \Lambda_1 W_1 = \Lambda_1/\mu_1$

Therefore:

$$\boldsymbol{L_1 = u_1}$$

By the forced flow law, this gives:

$$\boldsymbol{W_1^* = \frac{u_1}{\lambda_T}}$$

The transition state distribution of probabilities in the system is given by

$$\boldsymbol{P_1(n_1)} = \begin{cases} P(0) & \text{for } n=0 \\ (-1)^n \binom{-\theta}{n} \xi^n P(0) & \text{for } n=1,2,3, \ldots \end{cases}$$

where $\theta = \frac{\rho(1+C_{s1}^2)}{C_{a1}^2 - 1}$, $\xi = \frac{C_{a1}^2 - 1}{C_{a1}^2 + C_{s1}^2}$, $P(0) = (1-\xi)^{\theta}$

and $\binom{-\theta}{n} = \frac{(-1)^n \prod_{k=1}^{n}(\theta+k+1)}{n!} = (-1)^n \binom{\theta+n-1}{n}$, $n \geq 1$

2.2 Queuing Station II (GE/GE/1)

For the expected (mean) number of calls in the queuing station, we have

$$\mathbf{L_2} = \frac{\rho_2}{2}\left(1 + \frac{C_{a2}^2+\rho_2 C_{s2}^2}{1-\rho_2}\right)$$

By Little's law:

$$\mathbf{W_2 = L_2 \ / \ \Lambda_2}$$

and from the forced flow law, $W_2^* = W_2V_2$

where $V_2 = \Lambda_2 \ / \ \lambda_T = W_2 \ \Lambda_2 \ / \ \lambda_T$

then,

$$\mathbf{W_2^* = L_2 \ / \ \lambda_T}$$

The distribution of the state of probabilities for this model is given by:

$$P_2(n_2) = \begin{cases} 1-\rho_2 & \text{for } n=0 \\ (1-\rho_2)gx^n & \text{for } n=1,2,3,\ \ldots \end{cases}$$

where, $g = \frac{(1-x)\rho_2}{(1-\rho_2)x}$ and $x = \frac{L_2\rho_2}{L_2}$ ==> $P_2(n_2)=\rho_2(1-x)x^{n-1}$ for n = 1, 2, 3, ...

2.3 Queuing Station III (GE/GE/3)

The approximation of GE/GE/3 gives

$$\mathbf{L_3} = \frac{\rho_3}{2}\left(1 + \frac{C_{a3}^2+\rho_2 C_{s3}^2}{1-\rho_3}\right)$$

By Little's formula:

$$\mathbf{W_3 = L_3 \ / \ \Lambda_3}$$

and from the forced flow law, $W_3^* = W_3V_3$ where $V_3 = \Lambda_3 \ / \ \lambda_T = W_3 \ \Lambda_3 \ / \ \lambda_T$

yields:

$$\mathbf{W_3^* = L_3 \ / \ \lambda_T}$$

The distribution of probabilities when there are n_3 calls in the queuing station is given by:

$$P_3(n_3) = \begin{cases} \left[1 + \sum_{n=1}^{c=1} G_n + \frac{G_c}{1-x}\right]^{-1} & n = 0 \\ P(0)\left[\prod_{j=1}^{c} g_j{}^{hj(n)}\right] x^{L_{q(n)n}} & n = 1, 2, 3 \end{cases} \quad \text{where } G_n = \prod_{j=1}^{n} g_i \text{ n= 1, 2, 3}$$

and

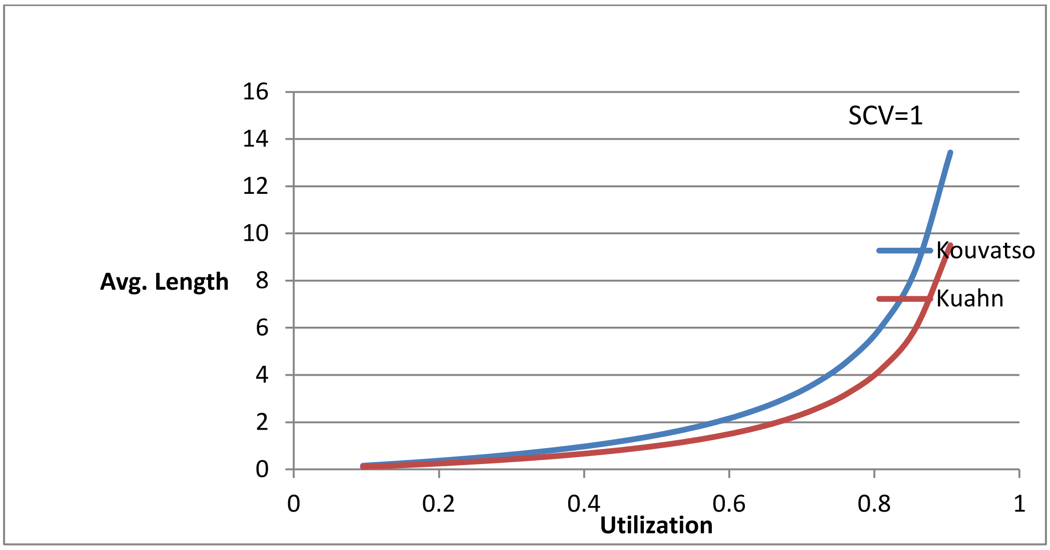
$$L_q = \sum_{n=j}^{\infty} P(n), \; j = 1, 2, 3$$

3.0 PART III: A PERFORMANCE COMPARATIVE STUDY

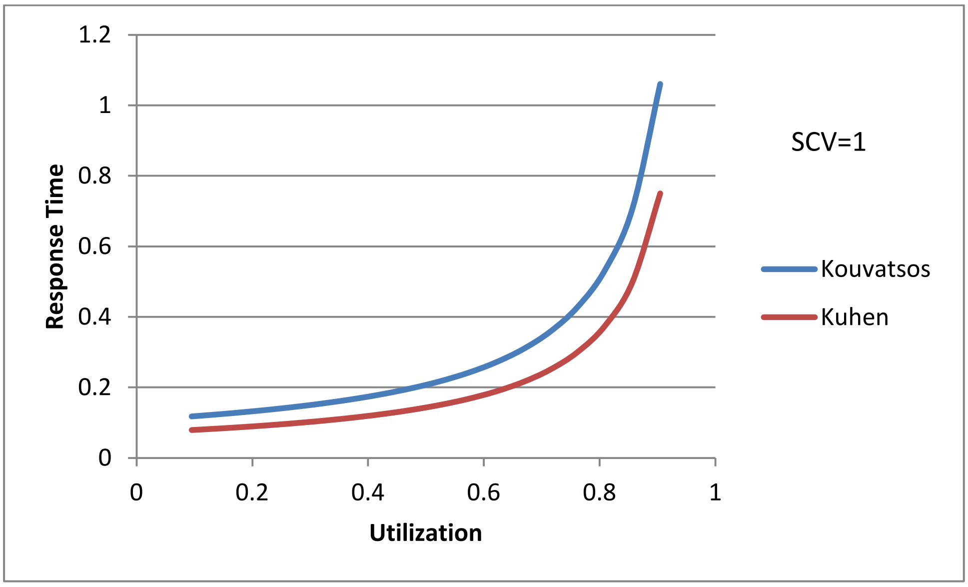
In an experiment to carry out a performance comparative study involving Jackson's open QNM, versus Kouvastos's open GE-type QNM and Kuehn's open QNM, different values of the square coefficient of arrival was chosen. The impact of this coefficient on performance was observed. The system behaves as Jackson's open QNM when the square coefficient of variance is 1.

Kouvatsos's Formula							
SCV=1							
Roh2	Roh3	L1	L2	L3	w1	w2	w3
0.095238	0.031746	0.071429	0.156799	0.048839	0.095238	0.1176	0.036629
0.142857	0.047619	0.071429	0.247421	0.074226	0.142857	0.12371	0.037113
0.190476	0.063492	0.071429	0.348126	0.100308	0.190476	0.130548	0.037615
0.238095	0.079365	0.071429	0.460823	0.127124	0.238095	0.138247	0.038137
0.285714	0.095238	0.071429	0.587925	0.154717	0.285714	0.146982	0.038679
0.333333	0.111111	0.071429	0.732536	0.183134	0.333333	0.156972	0.039243
0.380952	0.126984	0.071429	0.898714	0.212423	0.380952	0.168509	0.039829
0.428571	0.142857	0.071429	1.091869	0.242638	0.428571	0.181978	0.04044
0.47619	0.15873	0.071429	1.319377	0.273833	0.47619	0.197907	0.041075
0.523809	0.174603	0.071429	1.591567	0.306071	0.523809	0.217032	0.041737
0.571428	0.190476	0.071429	1.923354	0.339416	0.571428	0.240419	0.042427
0.619047	0.206349	0.071429	2.337112	0.373938	0.619047	0.269667	0.043147
0.666666	0.222222	0.071429	2.868	0.409715	0.666666	0.307286	0.043898
0.714285	0.238095	0.071429	3.574613	0.446828	0.714285	0.357462	0.044683
0.761904	0.253968	0.071429	4.562422	0.485365	0.761904	0.427727	0.045503
0.809523	0.269841	0.071429	6.042368	0.525425	0.809523	0.533151	0.046361
0.857142	0.285714	0.071429	8.506643	0.567113	0.857142	0.708888	0.047259
0.904761	0.301587	0.071429	13.43181	0.610542	0.904761	1.060407	0.048201

Table 1*: The performance when the SCV =1*



Graph 1: *The impact of square coefficient of variance on performance (SCV=1)*



Graph 2: *Response time when SCV= 1*

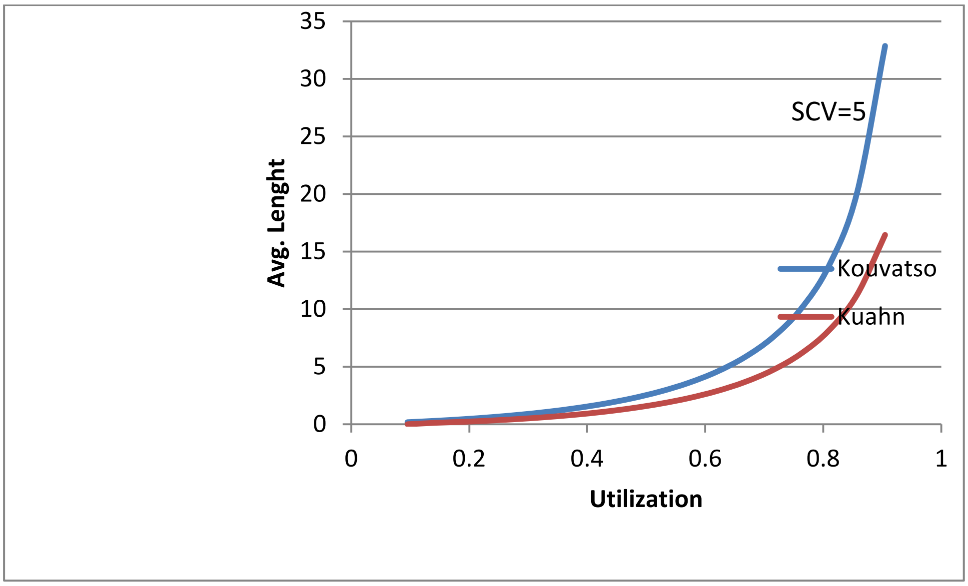
Kouvatsos's Formula							
SCV=5							
Roh2	Roh3	L1	L2	L3	w1	w2	w3
0.095238	0.031746	0.071429	0.177199	0.05103	0.095238	0.132899	0.038272
0.142857	0.047619	0.071429	0.296273	0.079358	0.142857	0.148137	0.039679
0.190476	0.063492	0.071429	0.440829	0.109801	0.190476	0.165311	0.041175

0.238095	0.079365	0.071429	0.615934	0.142546	0.238095	0.18478	0.042764
0.285714	0.095238	0.071429	0.827984	0.17779	0.285714	0.206996	0.044448
0.333333	0.111111	0.071429	1.085183	0.21574	0.333333	0.232539	0.04623
0.380952	0.126984	0.071429	1.398234	0.25661	0.380952	0.262169	0.048114
0.428571	0.142857	0.071429	1.781381	0.300625	0.428571	0.296897	0.050104
0.47619	0.15873	0.071429	2.254018	0.348019	0.47619	0.338103	0.052203
0.523809	0.174603	0.071429	2.843263	0.399041	0.523809	0.387718	0.054415
0.571428	0.190476	0.071429	3.588253	0.453951	0.571428	0.448532	0.056744
0.619047	0.206349	0.071429	4.547655	0.513023	0.619047	0.52473	0.059195
0.666666	0.222222	0.071429	5.81362	0.576551	0.666666	0.622889	0.061773
0.714285	0.238095	0.071429	7.53969	0.644845	0.714285	0.75397	0.064485
0.761904	0.253968	0.071429	10.00219	0.718237	0.761904	0.937706	0.067335
0.809523	0.269841	0.071429	13.75371	0.797084	0.809523	1.213564	0.070331
0.857142	0.285714	0.071429	20.08355	0.881769	0.857142	1.673631	0.073481
0.904761	0.301587	0.071429	32.85951	0.972705	0.904761	2.594174	0.076793

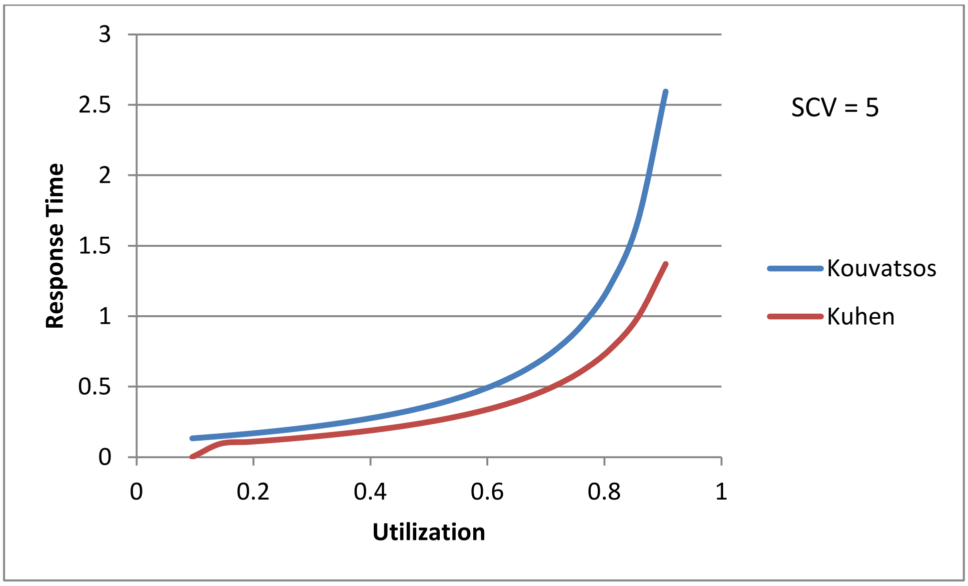
Table 2*: The impact of SCV on performance (SCV=5) according to Kouvatsos's formula*

Kuehn's Formula	SCV=5						
Roh2	Roh3	L1	L2	L3	w1	w2	w3
0.095238	0.031746	0.071429	0.125348	0.034879	0.095238	0.094011	0.02616
0.142857	0.047619	0.071429	0.214409	0.054799	0.142857	0.107205	0.0274
0.190476	0.063492	0.071429	0.32524	0.076495	0.190476	0.121965	0.028686
0.238095	0.079365	0.071429	0.461953	0.100068	0.238095	0.138586	0.030021
0.285714	0.095238	0.071429	0.629758	0.125626	0.285714	0.15744	0.031406
0.333333	0.111111	0.071429	0.835352	0.153283	0.333333	0.179004	0.032846
0.380952	0.126984	0.071429	1.087492	0.183162	0.380952	0.203905	0.034343
0.428571	0.142857	0.071429	1.397856	0.215397	0.428571	0.232976	0.0359
0.47619	0.15873	0.071429	1.782366	0.250129	0.47619	0.267355	0.037519
0.523809	0.174603	0.071429	2.263315	0.287513	0.523809	0.308634	0.039206
0.571428	0.190476	0.071429	2.872901	0.327712	0.571428	0.359113	0.040964
0.619047	0.206349	0.071429	3.659422	0.370906	0.619047	0.422241	0.042797
0.666666	0.222222	0.071429	4.698772	0.417287	0.666666	0.50344	0.044709
0.714285	0.238095	0.071429	6.117442	0.467064	0.714285	0.611745	0.046706
0.761904	0.253968	0.071429	8.143115	0.520463	0.761904	0.763418	0.048793
0.809523	0.269841	0.071429	11.23115	0.577731	0.809523	0.990985	0.050976
0.857142	0.285714	0.071429	16.44406	0.639135	0.857142	1.370339	0.053261
0.904761	0.301587	0.071429	26.96932	0.704967	0.904761	2.129159	0.055655

Table 3*: The impact of SCV on performance (SCV=5) using Kuehn's formula*



Graph 3: *The average number of customers when SCV=5*



Graph 4: *Response time when SCV= 5*

Kouvatsos's Formula	SCV=10						
Roh2	Roh3	L1	L2	L3	w1	w2	w3
0.095238	0.031746	0.071429	0.202695	0.053767	0.095238	0.152022	0.040325
0.142857	0.047619	0.071429	0.357322	0.085768	0.142857	0.178661	0.042884
0.190476	0.063492	0.071429	0.556638	0.121646	0.190476	0.208739	0.045617
0.238095	0.079365	0.071429	0.809602	0.161762	0.238095	0.242881	0.048529
0.285714	0.095238	0.071429	1.127496	0.206484	0.285714	0.281874	0.051621
0.333333	0.111111	0.071429	1.524749	0.256188	0.333333	0.326732	0.054897

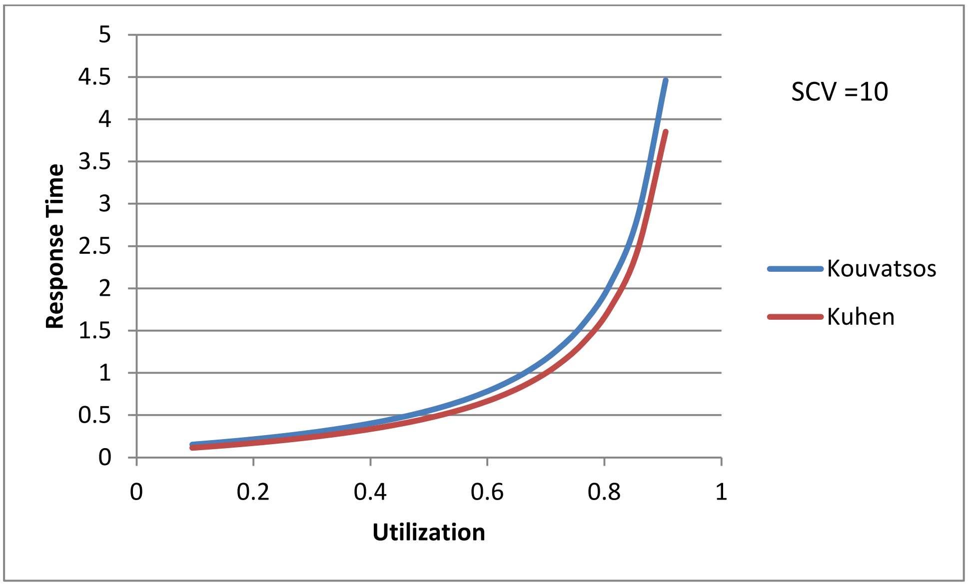
0.380952	0.126984	0.071429	2.020147	0.311256	0.380952	0.378778	0.058361
0.428571	0.142857	0.071429	2.638656	0.372083	0.428571	0.439776	0.062014
0.47619	0.15873	0.071429	3.414229	0.439073	0.47619	0.512135	0.065861
0.523809	0.174603	0.071429	4.394316	0.512645	0.523809	0.599225	0.069906
0.571428	0.190476	0.071429	5.64737	0.593235	0.571428	0.705922	0.074154
0.619047	0.206349	0.071429	7.275984	0.681303	0.619047	0.839537	0.078612
0.666666	0.222222	0.071429	9.441296	0.777332	0.666666	1.011568	0.083286
0.714285	0.238095	0.071429	12.41179	0.881835	0.714285	1.24118	0.088184
0.761904	0.253968	0.071429	16.67064	0.995363	0.761904	1.562874	0.093315
0.809523	0.269841	0.071429	23.18414	1.118506	0.809523	2.045661	0.098692
0.857142	0.285714	0.071429	34.20686	1.251901	0.857142	2.850575	0.104325
0.904761	0.301587	0.071429	56.50241	1.39624	0.904761	4.460721	0.11023

Table 4*: The impact of SCV on performance (SCV=10) using Kouvatsos's formula*

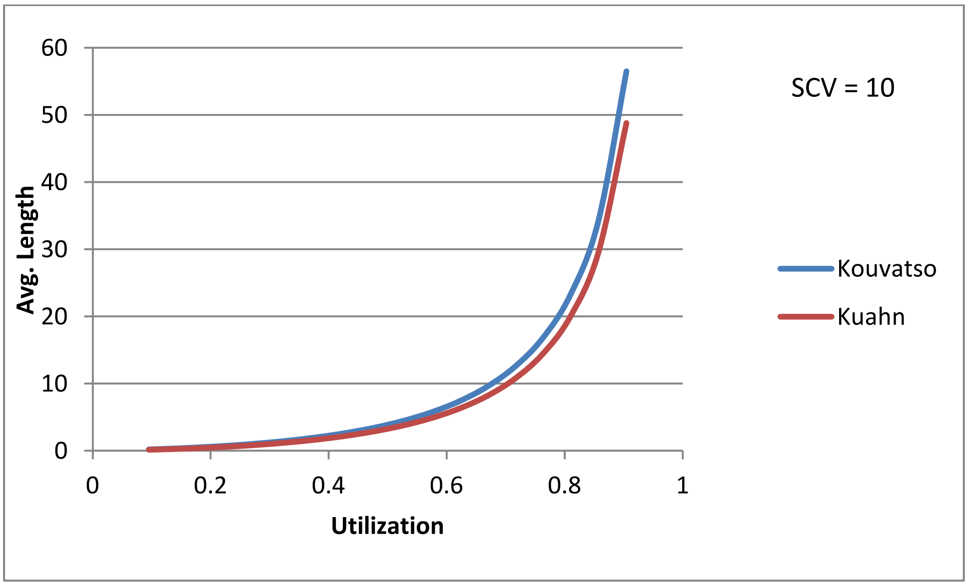
Kuehn's Formula	SCV=10						
Roh2	Roh3	L1	L2	L3	w1	w2	w3
0.095238	0.031746	0.071429	0.150454	0.037495	0.095238	0.112841	0.028121
0.142857	0.047619	0.071429	0.274088	0.060798	0.142857	0.137044	0.030399
0.190476	0.063492	0.071429	0.437673	0.087368	0.190476	0.164128	0.032763
0.238095	0.079365	0.071429	0.648771	0.117395	0.238095	0.194631	0.035219
0.285714	0.095238	0.071429	0.916956	0.151079	0.285714	0.229239	0.03777
0.333333	0.111111	0.071429	1.254542	0.188636	0.333333	0.268831	0.040422
0.380952	0.126984	0.071429	1.677627	0.230297	0.380952	0.314555	0.043181
0.428571	0.142857	0.071429	2.207677	0.27631	0.428571	0.367947	0.046052
0.47619	0.15873	0.071429	2.873963	0.326942	0.47619	0.431095	0.049041
0.523809	0.174603	0.071429	3.717463	0.38248	0.523809	0.506927	0.052156
0.571428	0.190476	0.071429	4.797365	0.443234	0.571428	0.599671	0.055404
0.619047	0.206349	0.071429	6.202455	0.509538	0.619047	0.715669	0.058793
0.666666	0.222222	0.071429	8.072244	0.581752	0.666666	0.864884	0.062331
0.714285	0.238095	0.071429	10.63926	0.660269	0.714285	1.063927	0.066027

0.761904	0.253968	0.071429	14.32203	0.74551	0.761904	1.342691	0.069892
0.809523	0.269841	0.071429	19.95762	0.837939	0.809523	1.760968	0.073936
0.857142	0.285714	0.071429	29.49918	0.938054	0.857142	2.458267	0.078171
0.904761	0.301587	0.071429	48.80608	1.046403	0.904761	3.853116	0.082611

Table 6*: The impact of SCV on performance (SCV=10) using Kuehn's formula*



Graph 5: *The response time when SCV= 10*



Graph 6: *The impact of SCV on performance (SCV=10)*

OBSERVATIONS

It was observed from the behaviour of the system that as the square coefficient of variance increases, the system performance deteriorates. This can be observed by the increase in the average length and the response time of the system as the square coefficient of variation increases from 1 to 10.

It was also observed that as the square coefficient of variation increases, the value obtained for Kuehn and Kouvastsos tends to overlap.

CONCLUSION

The system performance is best when the square coefficient of variance is at its minimum since the server will not be working at its full capacity.