Longitudinal waves and the mass-spring system

Introduction

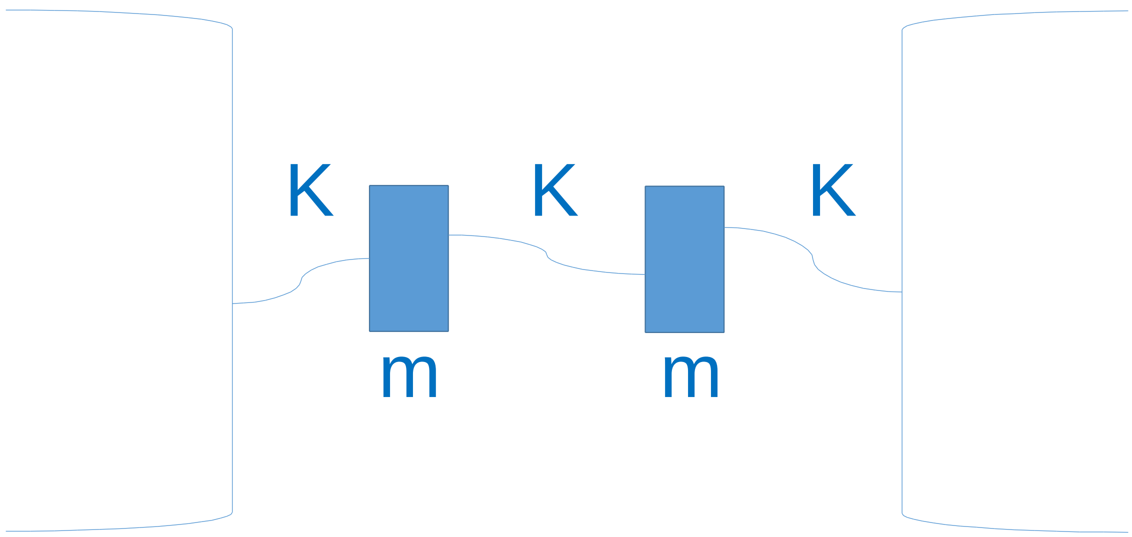
The mass-spring system is a simple but powerful model that provides insight into waves and their propagation. In particular, the equations describing the system can be simplified using elementary techniques and, in the appropriate limit, the wave equation can be found. Following identification of the wave equation, the form that waves can take is discussed.

This chapter presents the model in three sections. First, the case of two masses is analysed. Second, the case of arbitrarily-many masses is given attention. Finally, the case of infinitely-many masses, resulting in the wave equation, is studied. As everything is built up from first principles, there are few citations.

The two mass problem

For a single mass m on a spring with constant K, connected to a stationary wall and lying on a frictionless surface, there is one natural frequency, $\omega = \sqrt{K/m}$. Assuming neither damping nor driving forces, the displacement against time is sinusoidal. This problem should be familiar from discussions of simple harmonic motion (SHM).

A more sophisticated problem is the case of two identical masses m on a line, connected by identical springs with constant K to each other and each to a wall on their respective side. A diagram illustrating this is shown below. Again, there are neither driving nor damping forces, and gravity is assumed to be zero: the system concerns longitudinal displacements only.



Let x_1 and x_2 be the displacements of the respective masses from their equilibrium positions. For the sake of argument, it is assumed that in the equilibrium state of the system, the springs are all unstretched. By Hooke's law, the equations of motion of the two masses are

$$m\ddot{x}_1 = -Kx_1 + K(x_2 - x_1) = -K(2x_1 - x_2),$$

$$m\ddot{x}_2 = -Kx_2 - K(x_2 - x_1) = -K(2x_2 - x_1).$$

This is a system of coupled ordinary differential equations. There is more than one way it can be tackled, but all the most instructive methods work by decoupling the two equations in some way. The easiest way to do this is by introducing new variables so that each equation contains but one variable. For this problem, finding the variables can be done by inspection alone.

Take the new variables s$\varphi_1 = x_1 + x_2$, and $\varphi_2 = x_1 - x_2$. Rewriting the equations with the new variables leaves

$$m\ddot{\varphi}_1 = -K\varphi_1,$$

$$m\ddot{\varphi}_2 = -3K\varphi_2,$$

which are both equations for simple harmonic motion. Therefore, the solutions, which physically must be real-valued, have undetermined constants A_1, A_2, ϕ_1 and ϕ_2, and can be written as

$$\varphi_1 = 2A_1 \cos(\omega_1 t + \phi_1),$$

$$\varphi_2 = 2A_2 \cos(\omega_2 t + \phi_2),$$

where $\omega_1 = \sqrt{K/m}$ and $\omega_2 = \sqrt{3K/m}$.

These two independent solutions are called *normal modes*, and the variables φ_i are *normal coordinates* (Bajaj, 1984). Normal modes are, by definition, always sinusoidal and are examples of standing waves (Nettel, 2009). In this and other related problems, the most general solution for the system is a superposition of normal modes (Nettel, 2009): such a superposition is not a normal mode itself.

Finding the solution in terms of the original coordinates is straightforward. Recalling the defining equations and solving in terms of x_1 and x_2 gives

$$x_1 = \frac{\phi_1+\phi_2}{2} = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \text{ and}$$

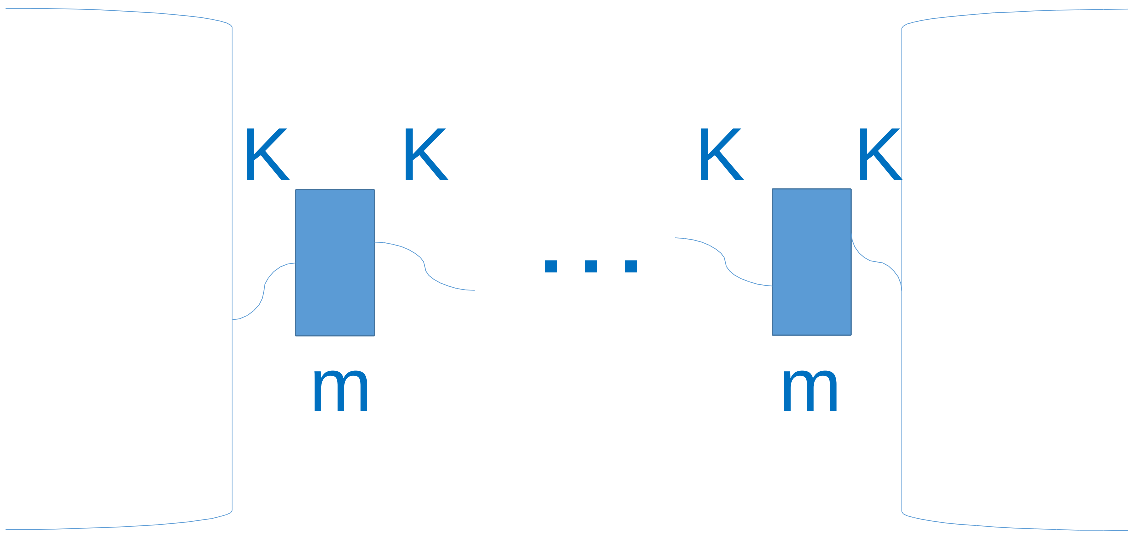
$$x_2 = \frac{\phi_1-\phi_2}{2} = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2).$$

If $A_2 = 0$, the two masses move sinusoidally in the same direction and the spring connecting them is unstretched. If $A_1 = 0$, the two masses move sinusoidally in opposite directions. If neither is zero, the motion of neither mass is sinusoidal.

This problem provides some insight into the wave-like behaviour of coupled oscillator systems, but the wave equation cannot be found in this system, and some additional understanding is required before getting to such a system.

The N mass problem

Consider the case of N masses m, connected as before but with more masses in between the two walls. The diagram below illustrates the set up.



The equations governing the displacement of the ith mass are straightforward. For $1 < i < N$, there are two springs attached to each mass, and their displacements from equilibrium are given as $x_{i+1}(t) - x_i(t)$ and $x_i(t) - x_{i-1}(t)$. This yields $\ddot{x}_i = \omega^2(x_{i+1} - x_i) - \omega^2(x_i - x_{i-1}) = \omega^2(x_{i+1} - 2x_i + x_{i-1})$, where $\omega = \sqrt{K/m}$. The first and last masses are governed by slightly different equations, these being $\ddot{x}_1 = \omega^2(x_2 - 2x_1)$ and $\ddot{x}_N = \omega^2(x_{N-1} - 2x_N)$.

Solving this system of coupled ordinary differential equations is ostensibly a difficult problem. In particular, unlike the last case, it is not obvious how to find the normal coordinates. Fortunately, it can be reduced to a much simpler problem with the machinery of matrix manipulation. First, the equations are written in matrix form,

$$\ddot{\boldsymbol{x}} = -\boldsymbol{\Omega}\boldsymbol{x}$$

where $\boldsymbol{\Omega}$ is an $N \times N$ matrix of the form

$$\boldsymbol{\Omega} = \omega^2 \begin{pmatrix} 2 & -1 & 0 & \cdots \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ \vdots & 0 & -1 & 2 \end{pmatrix} \text{and}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

There is no ‘best’ way to solve this system, but one of the most systematic ways is to use matrix diagonalisation. Consider:

$$\boldsymbol{\Omega} = \boldsymbol{A}\boldsymbol{\Omega}'\mathbf{A}^{-1}$$

and write

$$\boldsymbol{A}^{-1}\ddot{\boldsymbol{x}} = -\boldsymbol{\Omega}'\boldsymbol{A}^{-1}\boldsymbol{x}.$$

The next step is to introduce new coordinates, $\boldsymbol{\phi} = \boldsymbol{A}^{-1}\boldsymbol{x}$. These are the normal coordinates. The matrix $\boldsymbol{\Omega}'$ is the diagonal matrix of eigenvalues, and the eigenvalues correspond to the normal modes.

So, the system is now reduced to a series of uncoupled ODEs, those being

$$\ddot{\phi}_i = -\Omega'_{i,i}\phi_i.$$

Given that all the $\Omega'_{i,i}$ are real and positive, the normal modes are sinusoidal oscillations of the normal coordinates. As the equations for the time evolution of the normal coordinates are uncoupled, the most general solution to the problem is simply a superposition of normal modes, much as before. Returning to the original coordinates can be achieved by $\boldsymbol{x} = \boldsymbol{A}\boldsymbol{\phi}$.

As an explicit example, consider the system of three masses. The equations can be written as

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -\omega^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Diagonalisation yields

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -\omega^2 \begin{pmatrix} -1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2-\sqrt{2} & 0 \\ 0 & 0 & 2+\sqrt{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \\ \frac{1}{4} & \frac{-1}{2\sqrt{2}} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Writing

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{2} & 0 & \dfrac{1}{2} \\ \dfrac{1}{4} & \dfrac{1}{2\sqrt{2}} & \dfrac{1}{4} \\ \dfrac{1}{4} & \dfrac{-1}{2\sqrt{2}} & \dfrac{1}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

the equations are reduced to

$$\begin{pmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \\ \ddot{\varphi}_3 \end{pmatrix} = -\omega^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2-\sqrt{2} & 0 \\ 0 & 0 & 2+\sqrt{2} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix},$$

which are uncoupled linear equations, solved as

$$\varphi_1 = A_1 \cos(\omega_1 t + \phi_1),$$

$$\varphi_2 = A_2 \cos(\omega_2 t + \phi_2),$$

$$\varphi_3 = A_3 \cos(\omega_3 t + \phi_3),$$

where $\omega_1 = \sqrt{2}\omega$, $\omega_2 = \sqrt{2-\sqrt{2}}\omega$ and $\omega_3 = \sqrt{2+\sqrt{2}}\omega$.

Returning to the original coordinates yields

$$x_1 = -A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) + A_3 \cos(\omega_3 t + \phi_3),$$

$$x_2 = \sqrt{2}[A_2 \cos(\omega_2 t + \phi_2) - A_3 \cos(\omega_3 t + \phi_3)],$$

$$x_3 = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) + A_3 \cos(\omega_3 t + \phi_3).$$

As before, the normal coordinates move sinusoidally, but the original coordinates generally do not if more than one normal mode is excited.

Whilst this may seem no more wave-like than in the previous problem, it illustrates something important: the more distinct the masses, the more the normal modes and, furthermore, the more complex the functions that determine the time evolution of the original coordinates. Also, this case introduces the equations of time evolution of the central masses (those not connected to the walls).

Infinitely-many masses and the wave equation

The examples thus far illustrate some wave-like properties, such as sinusoidal oscillation, but the wave equation has yet to be identified. Enough of the background has been developed to progress to the next stage, the $N \to \infty$ problem, from where the wave equation is found. This case can be viewed as a model of an elastic substance.

The first step in dealing with the $N \to \infty$ problem is a change of notation. Let the x_n denote the equilibrium *positions*, and $\xi(x_n, t)$ denote the displacement *from* the equilibrium of the mass with equilibrium coordinate x_n. The equations determining the motion of the system (except for the first and last masses, which for simplicity are ignored here) are

$$\xi_{tt}(x_n, t) = \omega^2[\xi(x_{n-1}, t) - 2\xi(x_n, t) + \xi(x_{n+1}, t)].$$

Insight can be gained by another change of notation. The x_n can be replaced by a single number, x, the equilibrium position of the part of the system under consideration, which uniquely determines an x_n. Similarly, x_{n-1} can be written as $x - \Delta x$ and x_{n+1} as $x + \Delta x$, where Δx is the (equal) spacing between adjacent masses in equilibrium. Using the new coordinates it is found that

$$\xi_{tt}(x, t) = \omega^2[\xi(x - \Delta x, t) - 2\xi(x, t) + \xi(x + \Delta x, t)].$$

To progress further, an ostensible *regression* is needed. Writing $\omega^2 = K/m$ and manipulating it gives

$$m\xi_{tt}(x, t) = K[\xi(x - \Delta x, t) - 2\xi(x, t) + \xi(x + \Delta x, t)].$$

It is convenient to multiply both sides by $1/\Delta x$, with some manipulation of the right-hand side. This results in

$$\frac{m}{\Delta x}\xi_{tt}(x, t) = K\Delta x \frac{\frac{\xi(x-\Delta x,t) - 2\xi(x,t) + \xi(x+\Delta x,t)}{\Delta x}}{\Delta x},$$

$$\frac{m}{\Delta x}\xi_{tt}(x, t) = K\Delta x \frac{\frac{\xi(x+\Delta x,t) - \xi(x,t)}{\Delta x} - \frac{\xi(x,t) - \xi(x-\Delta x,t)}{\Delta x}}{\Delta x}.$$

The last line is in preparation for the next step: taking m and Δx to be infinitesimally small, and K to be infinite. If the latter assumption seems unreasonable, consider that halving the length of a spring doubles its spring constant. This leaves

$$\left(\frac{m}{\Delta x}\right)\xi_{tt}(x, t) = (K\Delta x)\frac{\xi_x(x,t) - \xi_x(x-\Delta x,t)}{\Delta x},$$

$$\left(\frac{m}{\Delta x}\right)\xi_{tt}(x, t) = (K\Delta x)\xi_{xx}(x, t).$$

Note that $\frac{m}{\Delta x}$ and $K\Delta x$ are finite.

Further simplification is possible. The quantity $\frac{m}{\Delta x}$ is the density ρ and the quantity $K\Delta x$ is the elastic modulus E. Therefore, a simpler form of the equation is

$$\rho\xi_{tt}(x,t) = E\xi_{xx}(x,t).$$

This is the wave equation (Nettel, 2009) and is one of the most important equations in mathematical physics. It is a partial differential equation, and most partial differential equations cannot be solved exactly. Fortunately, the wave equation is an exception.

Divide both sides of the equation by ρ, and construct a new variable $v = E/\rho$. The equation is now

$$\xi_{tt}(x,t) = v\xi_{xx}(x,t).$$

Factoring the equation is one way to solve it. Write

$$0 = \xi_{tt}(x,t) - v\xi_{xx}(x,t)$$

followed by

$$0 = (\partial_t - v\partial_x)(\partial_t + v\partial_x)\,\xi(x,t).$$

The two solutions are found by

$$0 = (\partial_t - v\partial_x)\xi(x,t),$$

$$0 = (\partial_t + v\partial_x)\,\xi(x,t),$$

leading to

$$\xi(x,t) = f(x+vt),$$

$$\xi(x,t) = g(x-vt),$$

where f and g are twice-differentiable functions. As the equation is linear, a superposition of solutions is a solution itself. f is a function with a displacement moving to the left with time, g is a function with a displacement moving to the right with time. Both move with velocity v in their respective directions. Note that, unlike the finite-N case, there are no restrictions on what form the solutions take other than twice-differentiability.

It is noted that the finite length of the mass-spring system places a limit on the domain of applicability of the wave equation. Specifically, it cannot hold at the ends, whether they are constrained by walls or not.

Familiar sorts of waves, like trigonometric functions, can be solutions to the wave equation, and a superposition of solutions can result in a standing wave, like those observed in the finite N problem. For example, if $f(x+vt) = \cos(x+vt)$ and $g(x-vt) = \cos(x-vt)$, the complete solution is, when simplified, $\xi(x,t) = 2\cos(x)\cos(vt)$, a standing wave.

References

Nettel, S., 2009. *Wave Physics: Oscillations - Solitons – Chaos*. 4th ed. Springer.

Bajaj, N. K., 1984. *The Physics of Waves and Oscillations*. New Delhi: Tata McGraw-Hill Education.